TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, February 1970

MEAN CONVERGENCE OF HERMITE AND LAGUERRE SERIES. I

BY
BENJAMIN MUCKENHOUPT[1]

1. **Introduction.** In [4], Pollard showed that there is an inequality of the type,

$$\int_{-\infty}^{\infty} |s_n(x)|^p e^{-x^2}\,dx \leqq C\int_{-\infty}^{\infty} |f(x)|^p e^{-x^2}\,dx,$$

where $s_n(x)$ is the nth partial sum of the Hermite polynomial series for $f(x)$, only for $p=2$. This, of course, implies that $\lim_{n\to\infty}\int_{-\infty}^{\infty} |s_n(x)-f(x)|^p e^{-x^2}\,dx=0$ for every $f(x)$ satisfying $\int_{-\infty}^{\infty} |f(x)|^p e^{-x^2}dx<\infty$ only in case $p=2$. With $\|\ \ \|_p$ denoting the usual (unweighted) norm on $(-\infty,\infty)$, Askey and Wainger in [1] showed that there is an inequality of the type, $\|s_n(x)e^{-x^2/2}\|_p\leqq C\|f(x)e^{-x^2/2}\|_p$, for $4/3<p<4$ and not for other values of p, and they obtained the corresponding mean convergence theorem. Similar results were proved in both papers for Laguerre series.

The contrast of these results and the fact that trigonometric series converge in the mean for $1<p<\infty$ suggest that an inequality of the form $\|s_n(x)w(x)\|_p \leqq C\|f(x)w(x)\|_p$ should be possible for $p\geqq 4$ or $1<p\leqq 4/3$ if $w(x)$, possibly depending on p, is chosen properly. The main results of this paper are that not even the individual terms of a Hermite or Laguerre series can satisfy such an inequality and that the terms will not converge to 0 in the mean if p is not between 4/3 and 4. Specifically, the following will be proved.

THEOREM 1. *Let p be a fixed number satisfying $1\leqq p\leqq 4/3$ or $4\leqq p\leqq\infty$, let $\|\ \ \|_p$ denote the ordinary (unweighted) norm on $(-\infty,\infty)$, let $w(x)$ be finite almost everywhere on $(-\infty,\infty)$, let $f(x)$ be a function which has a Hermite series and satisfies $\|w(x)f(x)\|_p<\infty$, and let $a_nH_n(x)$ be the nth term of that series. If there exists a constant, C, independent of $f(x)$, such that $\|a_nH_n(x)w(x)\|_p\leqq C\|f(x)w(x)\|_p$ for all $n\geqq C$, then $w(x)=0$ almost everywhere.*

THEOREM 2. *Let p, $\|\ \ \|_p$, $w(x)$, $f(x)$ and a_n be as in the first sentence of Theorem 1. If $\lim_{n\to\infty}\|a_nH_n(x)w(x)\|_p=0$ for every such $f(x)$, then $w(x)=0$ almost everywhere.*

THEOREM 3. *Let p be a fixed number satisfying $1\leqq p\leqq 4/3$ or $4\leqq p\leqq\infty$, let $\|\ \ \|_p$ denote the ordinary (unweighted) norm on $(0,\infty)$, let α be a fixed number greater than -1, let $w(x)$ be finite almost everywhere on $(0,\infty)$, let $f(x)$ be a function*

Presented to the Society, January 23, 1969; received by the editors November 7, 1968.
[1] Supported in part by NSF grant GP 7539.

which has a Laguerre series for this α *and satisfies* $\|w(x)f(x)\|_p<\infty$, *and let* $a_nL_n^\alpha(x)$ *be the nth term of that series. If there exists a constant, C, independent of* $f(x)$ *such that* $\|a_nL_n^\alpha(x)w(x)\|_p\leqq C\|f(x)w(x)\|_p$ *for all* $n\geqq C$, *then* $w(x)=0$ *almost everywhere.*

THEOREM 4. *Let* p, $\| \quad \|_p$, $w(x)$, $f(x)$, α, *and* a_n *be as in the first sentence of Theorem* 3. *If* $\lim_{n\to\infty}\|a_nL_n^\alpha(x)w(x)\|_p=0$ *for every such* $f(x)$, *then* $w(x)=0$ *almost everywhere.*

The following are immediate corollaries by Minkowski's inequality.

COROLLARY 1. *Let* p, $\| \quad \|_p$, $w(x)$ *and* $f(x)$ *be as in the first sentence of Theorem* 1 *and let* $s_n(x)$ *be the nth partial sum of f's Hermite series. If* $\lim_{n\to\infty}\|[s_n(x)-f(x)]w(x)\|_p=0$ *for every such* $f(x)$, *then* $w(x)=0$ *almost everywhere.*

COROLLARY 2. *Let* p, $\| \quad \|_p$, $w(x)$, $f(x)$ *and* α *be as in the first sentence of Theorem* 3 *and let* $s_n(x)$ *be the nth partial sum of f's Laguerre series. If*

$$\lim_{n\to\infty}\|[s_n(x)-f(x)]w(x)\|_p=0$$

for every such $f(x)$, *then* $w(x)=0$ *almost everywhere.*

The requirement that $w(x)$ be finite almost everywhere is put in these theorems to avoid uninteresting complications in the proofs. With the convention $0\cdot\infty=0$, $w(x)\equiv\infty$ would satisfy the conditions in each of these since $w(x)f(x)\in L^p$ would require that $f(x)=0$ almost everywhere. If $w(x)$ is finite on a subset of positive measure and infinite on a subset of positive measure of the interval under consideration, it is immediate that it cannot satisfy the hypotheses of any of these theorems.

Theorem 1 will be proved in four parts. The first is Lemma 1 in which it is shown that if a function, $w(x)$, of the type described in Theorem 1, existed that was not 0 almost everywhere, then there would be a function, $v(x)$, such that for $n\geqq C$

$$\|H_n(x)e^{-x^2/2}v(x)\|_p\left\|\frac{H_n(x)e^{-x^2/2}}{v(x)}\right\|_q \leqq C\|H_n(x)e^{-x^2/2}\|_2^2 \tag{1.1}$$

where $1/p+1/q=1$ and $H_n(x)$ is the usual Hermite polynomial.

The second part, contained in §4, consists of showing that the terms, $H_n(x)e^{-x^2/2}$, on the left side of (1.1) can be replaced over a limited range by a simple estimate and the inequality will still be true. The principal difficulty in doing this is the fact that the range considered is where $H_n(x)$ has its zeros. The proof amounts to showing that successive H_n's have their zeros well enough distributed so that for most n's the integrals on the left will be bounded below by their estimates. An estimate in [2] of $H_n(x)$, proved by Skovgaard, is the basis for this proof.

In §5 it is proved that an inequality like the simplified version of (1.1) cannot be true for any $v(x)$. This follows from estimations of the integrals, Hölder's inequality and Fubini's theorem. Finally, in §6 these results are combined to prove Theorem 1.

Theorem 2 would be an immediate consequence of Theorem 1 and the Banach-Steinhaus theorem if it were assumed that $w(x)f(x) \in L^p$ implied that $f(x)$ had a Hermite series. In §7 it is shown that if $w(x)$ satisfies the hypotheses of Theorem 2 but not the conclusion, then there is a function, $w^*(x)$, such that $|w(x)| < w^*(x)$, $w^*(x)$ satisfies the hypotheses of Theorem 2 and $w^*(x)f(x) \in L^p$ implies that $f(x)$ has a Hermite series. Theorem 1 and the Banach-Steinhaus theorem then imply that $w^*(x)$ cannot exist and thus prove Theorem 2.

The proof of Theorems 3 and 4 is similar and is treated simultaneously.

In the sequel to this paper, inequalities of the form $\|s_n(x)v(x)\|_p \leqq C\|f(x)w(x)\|_p$ will be proved for Laguerre and Hermite series for $1 \leqq p \leqq 4/3$ and $4 \leqq p \leqq \infty$.

2. **Notation and estimates.** To simplify notation, two functions introduced in [1] will be used. The Hermite polynomials, $H_n(x)$, are defined by $\sum H_n(x)r^n/n! = \exp(2xr - r^2)$; the functions

$$\mathscr{H}_n(x) = e^{-x^2/2}(\sqrt{\pi}2^n n!)^{-1/2}H_n(x) \tag{2.1}$$

are orthonormal on $(-\infty, \infty)$. Similarly, the Laguerre polynomials, $L_n^\alpha(x)$ are defined by $\sum L_n^\alpha(x)r^n = (1-r)^{-\alpha-1}\exp(-rx/(1-r))$; the functions

$$\mathscr{L}_n^\alpha(x) = \left[\frac{\Gamma(n+\alpha+1)}{n!}\right]^{-1/2} x^{\alpha/2}e^{-x/2}L_n^\alpha(x) \tag{2.2}$$

are orthonormal on $[0, \infty)$.

Now (6.12), p. 23 of [2] states that $H_n(N^{1/2}\cos\theta)$ equals

$$\left(\frac{2N^n}{\sin\theta}\right)^{1/2} \exp\left(\frac{N\cos 2\theta}{4}\right)\left[\cos\left(\frac{N(2\theta - \sin 2\theta) - \pi}{4}\right) + O\left(\frac{1}{n\theta^3}\right)\right] \tag{2.3}$$

for $0 < \theta \leqq \frac{1}{2}\pi$ where $N = 2n+1$. Combining (2.1) and (2.3) and using Stirling's formula then shows that

$$\mathscr{H}_n(x) = \left(\frac{2}{\pi}\right)^{1/2}(N-x^2)^{-1/4}\cos\left(\frac{N(2\theta - \sin 2\theta) - \pi}{4}\right) + O(N^{1/2}(N-x^2)^{-7/4}) \tag{2.4}$$

where $0 \leqq x \leqq N^{1/2} - N^{-1/6}$ and $\theta = \cos^{-1}(xN^{-1/2})$. Furthermore, using either (2.4) or combining (8.22.8), p. 198 of [5] with (2.1) shows that for x in a fixed finite interval

$$\mathscr{H}_n(x) = \left(\frac{2}{\pi}\right)^{1/2} N^{-1/4}\cos(N^{1/2}x - \tfrac{1}{2}n\pi) + O(N^{-3/4}). \tag{2.5}$$

Finally, the table on p. 700 of [1] shows that there exist positive constants, C and D, such that

$$\begin{aligned} |\mathscr{H}_n(x)| &\leqq C[|N-x^2| + N^{1/3}]^{-1/4}, & x^2 &< N, \\ &\leqq C\exp(-Dx^2), & x^2 &\geqq N. \end{aligned} \tag{2.6}$$

Similarly, (5.5), p. 247 of [3] states that if $\alpha \geqq 0$, then $L_n^\alpha(\nu \cos^2 \theta)$ equals

$$(2.7) \quad \frac{(-1)^n \exp(\frac{1}{2}\nu \cos^2 \theta)}{(2 \cos \theta)^\alpha (\pi n \sin 2\theta)^{1/2}} \left[\cos\left(\frac{\nu(2\theta - \sin 2\theta) - \pi}{4}\right) + O\left(\frac{1}{\nu\theta^3} + \frac{1}{\nu(\frac{1}{2}\pi - \theta)}\right)\right]$$

where $0 < \theta < \frac{1}{2}\pi$ and $\nu = 4n + 2\alpha + 2$. Again, combining (2.2) and (2.7) and using Stirling's formula gives

$$(2.8) \quad \mathscr{L}_n^\alpha(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^n}{x^{1/4}(\nu - x)^{1/4}} \cos\left(\frac{\nu(2\theta - \sin 2\theta) - \pi}{4}\right) + O\left(\frac{\nu^{1/4}}{(\nu - x)^{7/4}} + (\nu x)^{-3/4}\right)$$

where $0 < x < \nu$, $\alpha \geqq 0$ and $\theta = \cos^{-1}(x^{1/2}\nu^{-1/2})$. Fortunately, (2.7) is also valid for $-1 < \alpha < 0$ for a more restricted range of x; this fact, proved in the next paragraph, will be needed for the proofs of Theorems 3 and 4.

Using the fact, (5.1.13), p. 101 of [5], that $L_n^\alpha(x) = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x)$, it is easy to prove that

$$(2.9) \quad x^{1/2}\mathscr{L}_n^\alpha(x) = (n + \alpha + 1)^{1/2}\mathscr{L}_n^{\alpha+1}(x) - n^{1/2}\mathscr{L}_{n-1}^{\alpha+1}(x).$$

Using (2.8) in (2.9), replacing $n + \alpha + 1$ and n by $\frac{1}{4}\nu + O(1)$ and letting $p(\nu, x)$ denote the principal term in (2.8), then shows that

$$(2.10) \quad \nu^{1/2}p(\nu + 2, x) - \nu^{1/2}p(\nu - 2, x) = 2x^{1/2}p(\nu, x) + O\left(\frac{\nu^{3/4}}{(\nu - x)^{7/4}} + \frac{1}{\nu^{1/4}x^{3/4}}\right)$$

for $0 < x \leqq \nu - 4$ and $\alpha \geqq 0$. Since α does not appear in (2.10), (2.10) is clearly true with just the first of these conditions. Using (2.8) on the right side of (2.9) for $-1 < \alpha < 0$ and then applying (2.10) proves that

$$(2.11) \quad \mathscr{L}_n^\alpha(x) = p(\nu, x) + O\left(\frac{\nu^{1/4}}{(\nu - x)^{7/4}} + \frac{1}{\nu^{1/4}x^{5/4}}\right)$$

for $-1 < \alpha < 0$ and $0 < x \leqq \nu - 4$. This gives (2.8) if $-1 < \alpha < 0$ and $\frac{1}{2}\nu \leqq x \leqq \nu - 4$; this range of x is sufficient for the purposes of this paper.

Combining (8.22.6), p. 197 of [5] with (2.2) shows that for x in a fixed compact subinterval of $(0, \infty)$ and $\alpha > -1$

$$(2.12) \quad \mathscr{L}_n^\alpha(x) = \left(\frac{2}{\pi}\right)^{1/2} (x\nu)^{-1/4} \cos\left[(\nu x)^{1/2} - \tfrac{1}{2}\alpha\pi - \tfrac{1}{4}\pi\right] + O(\nu^{-3/4}).$$

The table on p. 699 of [1] shows that if $\alpha \geqq 0$, then there exist positive constants, C and D, such that

$$(2.13) \quad \begin{aligned} |\mathscr{L}_n^\alpha(x)| &\leqq C(nx)^{\alpha/2}, && 0 \leqq x < 1/n, \\ &\leqq Cx^{-1/4}[|\nu - x| + \nu^{1/3}]^{-1/4}, && 1/n \leqq x < 2\nu, \\ &\leqq Ce^{-Dx}, && 2\nu \leqq x. \end{aligned}$$

It is easy to show that (2.13) is also true for $-1 < \alpha < 0$. For $0 \leqq x < 1/n$ the explicit form of $L_n^\alpha(x)$, (5.1.6), p. 100 of [5], gives the result. For $1/n \leqq x < 1$, (8.22.6) on

p. 197 of [5] can be used. For $1 \leqq x < n$, use (2.11). For $n \leqq x$, (2.9) can be used to obtain the result.

Throughout this paper the symbol $\| \quad \|_p$ will designate the ordinary (unweighted) norm over the interval $(-\infty, \infty)$ in the Hermite case and over $(0, \infty)$ in the Laguerre case. The letter C will be used to denote positive constants not necessarily the same at each occurrence.

3. **An integral inequality.** Part of the following lemma is a generalization of an argument presented on p. 706 of [1].

LEMMA 1. *If p, $w(x)$, $f(x)$, a_n and C satisfy the hypotheses of Theorem 1 and $w(x)$ is nonzero on a set of positive measure, then there exists a function $v(x)$, such that for all $n \geqq C$*

$$\|\mathscr{H}_n(x)v(x)\|_p \, \|\mathscr{H}_n(x)/v(x)\|_q \leqq C \tag{3.1}$$

where $1/p+1/q=1$. The same is true with the hypotheses of Theorem 3 if $\mathscr{H}_n(x)$ is replaced by $\mathscr{L}_n^\alpha(x)$ in (3.1).

For the Hermite case it will first be proved that if $w(x)$ has the given properties and $g(x)w(x) \in L^p$, then $g(x)$ must have a Hermite expansion. If there were a $g(x)$ such that $g(x)w(x) \in L^p$ and $g(x)$ had no Hermite expansion, then for some N

$$\|g(x) \exp(-x^2)x^N\|_1 = \infty.$$

If $N=0$, then for all even n $\|g(x)H_n(x) \exp(-x^2)\|_1 = \infty$; if $N \geqq 1$, then for all $n \geqq N$ this integral is infinite. Now let $h(x)$ be a bounded function with compact support such that $|h(x)| \leqq |g(x)|$ and let $b_n H_n(x)$ be the nth term of h's Hermite series. Then for $n \geqq C$

$$|b_n| \, \|w(x)H_n(x)\|_p \leqq C\|w(x)h(x)\|_p \leqq C\|w(x)g(x)\|_p. \tag{3.2}$$

The hypothesis that $w(x)$ is nonzero on a set of positive measure insures that $\|w(x)H_n(x)\|_p > 0$. If n is even and greater than N, $h(x)$ can be chosen to make b_n arbitrarily large; this contradicts (3.2).

Now if $g(x)$ is any function in L^p, the result above shows that $(g(x)/w(x))H_n(x)e^{-x^2}$ is in L^1 for every n. Consequently, by the converse of Hölder's inequality

$$\|\exp(-x^2)H_n(x)/w(x)\|_q < \infty \tag{3.3}$$

for all n.

Let $r_n = (\sqrt{\pi}2^n n!)^{-1/2}$, then $a_n = r_n^2 \int_{-\infty}^{\infty} f(x)H_n(x) \exp(-x^2)\, dx$. For a fixed n there is a function, $f(x)$, such that $w(x)f(x) \in L^p$ and

$$|a_n| = r_n^2 \|w(x)f(x)\|_p \|\exp(-x^2)H_n(x)/w(x)\|_q$$

because of (3.3). If $n \geqq C$, it is also true that $|a_n| \, \|w(x)H_n(x)\|_p \leqq C\|f(x)w(x)\|_p$. Combining these two facts and using the fact that $H_n(x) = \exp(x^2/2)\mathscr{H}_n(x)/r_n$ gives (3.1) with $v(x) = w(x) \exp(x^2/2)$.

The Laguerre proof is the same.

4. **Estimation of the integrals.** The purpose here is to obtain lower bounds for the terms on the left side of (3.1) that contain pleasanter functions than $\mathscr{H}_n(x)$ or $\mathscr{L}_n^\alpha(x)$. The first part of this consists of looking at the troublesome cosine terms in the approximations (2.4) and (2.8). The computation is based on the following rather interesting lemma.

LEMMA 2. *Let L be an integer greater than* 20 *and let I be a set of L consecutive integers. If for n in I,* $1/3L \leqq g(n+1)-g(n) \leqq \frac{1}{4}\pi$ *and* $g(n+1)-g(n)$ *is monotone increasing in n, then for at least* 2/3 *of the integers, n, in I,* $|\cos g(n)| \geqq 1/200$.

The intervals in which $|\cos x| < 1/200$ have length $2 \sin^{-1}(1/200)$ which is less than 1/90. The first of these intervals that contains any $g(n)$'s then contains at most $[L/30]+1$ of them where [ ] denotes the greatest integer. After this, each interval where $|\cos x| \geqq 1/200$ will contain at least three times as many $g(n)$'s as the succeeding interval where $|\cos x| < 1/200$ because of the upper bound on $g(n+1)-g(n)$ and the monotonicity of $g(n+1)-g(n)$. Then for at most $[L/30]+1+\frac{1}{4}(L-[L/30]-1)$ of the n's in I is $|\cos g(n)| < 1/200$. Since $(1/L)[L/30] \leqq 1/30$, this number is bounded by $L(1/40+\frac{1}{4}+3/4L)$ which is less than $L/3$ since $L > 20$.

This can be applied to the following two lemmas.

LEMMA 3. *If* $y \geqq (60)^3$ *and x is a fixed number such that* $\frac{3}{4}y^{1/2} \leqq x \leqq y^{1/2}-y^{-1/6}$, *then for at least two thirds of the integers, n, such that* $y \leqq 2n+1 \leqq y+y^{1/3}$,

$$\left|\cos\left(\frac{N(2\theta-\sin 2\theta)-\pi}{4}\right)\right| > \frac{1}{200}$$

where $N=2n+1$ *and* $\theta=\cos^{-1}(xN^{-1/2})$.

LEMMA 4. *If* $y \geqq (90)^3$, $\alpha > -1$ *and x is a fixed number such that* $5y/6 \leqq x \leqq y-y^{1/3}$, *then for at least two thirds of the integers, n, such that* $y \leqq 4n+2\alpha+2 \leqq y+y^{1/3}$,

$$\left|\cos\left(\frac{\nu(2\theta-\sin 2\theta)-\pi}{4}\right)\right| \geqq \frac{1}{200}$$

where $\nu=4n+2\alpha+2$ *and* $\theta=\cos^{-1}(x^{1/2}\nu^{-1/2})$.

The proofs are similar and simple. For Lemma 3 the number, L, of consecutive integers, n, with $y \leqq 2n+1 \leqq y+y^{1/3}$ satisfies $y^{1/3}/3 \leqq L \leqq y^{1/3}$. Define

$$g(n) = (N(2\theta-\sin 2\theta)-\pi)/4.$$

It is easily verified that $g'(n)=\cos^{-1}(xN^{-1/2})$. If x is in the given range and n is any real number in the extended range, $y \leqq 2n+1 \leqq y+y^{1/3}+2$, then

$$2^{-1/2} \leqq \tfrac{3}{4}y^{1/2}(y+y^{1/3}+2)^{-1/2} \leqq xN^{-1/2} \leqq 1-y^{-2/3}. \tag{4.1}$$

From this it is clear that $\cos^{-1}(xN^{-1/2})$ is monotone increasing in n and bounded above by $\frac{1}{4}\pi$; this immediately implies the same for $g(n+1)-g(n)$ if $y \leqq 2n+1 \leqq y+y^{1/3}$.

Now if $\cos u \leqq 1-y^{-2/3}$, then $1-\frac{1}{2}u^2 \leqq 1-y^{-2/3}$ and $u \geqq 2^{1/2}y^{-1/3}$. Therefore, using (4.1), $g'(n) \geqq 2^{1/2}y^{-1/3} \geqq 2^{1/2}/3L$ for the extended range of n's used in (4.1). This implies that $g(n+1)-g(n) \geqq 1/3L$ for x and n in the original ranges. Lemma 2 now completes the proof of Lemma 3.

The proof of Lemma 4 proceeds in the same manner.

To apply these results to the integrals in (3.1) the following general lemma is needed.

LEMMA 5. *Let $w(x)$ be a nonnegative function and t a positive real number. Let $f(n, x)$ be a function such that for every x in a set E and every integer in a finite set of integers, I, $0 \leqq f(n, x) \leqq 1$, and for each x in E, $f(n, x) \geqq t$ for at least $2/3$ of the n's in I. Then $\int_E f(n, x)w(x)\,dx \geqq (t/10)\int_E w(x)\,dx$ for at least $3/5$ of the n's in I.*

Let $f^*(n, x)=1$ if $f(n, x) \geqq t$ and let $f^*(n, x)=0$ otherwise. Then

$$\sum_{n\in I}\int_E f^*(n, x)w(x)\,dx \geqq \frac{2L}{3}\int_E w(x)\,dx \tag{4.2}$$

where L is the number of members of I. Then for at least $3/5$ of the n's in I

$$\int_E f^*(n, x)w(x)\,dx \geqq \frac{1}{10}\int_E w(x)\,dx; \tag{4.3}$$

this follows from (4.2), the fact that the integral on the left is always bounded by the integral on the right and the fact that $3L/5+(2L/5)1/10<2L/3$. Since $f(n, x) \geqq tf^*(n, x)$, (4.3) gives the conclusion of the lemma.

LEMMA 6. *Let $w(x)$ be a measurable function, $1 \leqq p \leqq \infty$, for $y>0$ let $E_y=[\frac{3}{4}y^{1/2}, y^{1/2}-1]$ and I_y be the integers, n, such that $y \leqq N \leqq y+y^{1/3}$ where $N=2n+1$. Then there exists y_0 and positive C depending only on p such that for $y \geqq y_0$*

$$\left(\int_{E_y}|\mathscr{H}_n(x)w(x)|^p\,dx\right)^{1/p} \geqq C\left(\int_{E_y}\left|\frac{w(x)}{(y-x^2)^{1/4}}\right|^p dx\right)^{1/p} \tag{4.4}$$

(with the usual interpretation for $p=\infty$) for at least $3/5$ of the n's in I_y.

LEMMA 7. *Let $w(x)$ be a measurable function, $1 \leqq p \leqq \infty$, $\alpha>-1$, for $y>1$ let $E_y=[5y/6,\ y-y^{1/2}]$ and I_y the integers, n, such that $y \leqq \nu \leqq y+y^{1/3}$ where $\nu=4n+2\alpha+2$. Then there exists y_0 and positive C depending only on p such that for $y \geqq y_0$*

$$\left(\int_{E_y}|\mathscr{L}_n^\alpha(x)w(x)|^p\,dx\right)^{1/p} \geqq C\left(\int_{E_y}\left|\frac{w(x)}{x^{1/4}(y-x)^{1/4}}\right|^p dx\right)^{1/p} \tag{4.5}$$

(with the usual interpretation for $p=\infty$) for at least $3/5$ of the n's in I_y.

To prove Lemma 6, observe that $N-x^2 \geqq y-x^2$, and since $y-x^2 \geqq 2y^{1/2}-1 \geqq y^{1/3}$, then $N-x^2 \leqq 2(y-x^2)$. Using these inequalities, (2.4) and Minkowski's inequality, shows that the left side of (4.4) is bounded below by the difference of

$$C_1\left(\int_{E_y}\left|\frac{w(x)}{(y-x^2)^{1/4}}\cos\left(\frac{N(2\theta-\sin 2\theta)-\pi}{4}\right)\right|^p dx\right)^{1/p} \tag{4.6}$$

and

$$C_2\left(\int_{E_y}\left|\frac{y^{1/2}w(x)}{(y-x^2)^{7/4}}\right|^p dx\right)^{1/p} \tag{4.7}$$

where C_1 and C_2 are positive constants and $\theta=\cos^{-1}(xN^{1/2})$.

Applying Lemmas 3 and 5 to (4.6) shows that there exists a $C_3>0$ such that if $y\geqq(60)^3$, then (4.6) is bounded below by C_3 times the integral on the right side of (4.4) for at least 3/5 of the n's in I_y. On the other hand, since $y-x^2\geqq y^{1/2}$ for x in E_y, (4.7) is bounded above by $C_2y^{-1/4}$ times the integral on the right side of (4.4). If y is greater than $(2C_2/C_3)^4$ and $(60)^3$, then (4.7) is less than half of (4.6) for at least 3/5 of the n's in I_y and the assertion of Lemma 6 is proved.

Lemma 7 is proved similarly using (2.8) and Lemma 4.

5. **An integration lemma.** This section is devoted to showing that an inequality like (3.1) cannot occur if $\mathscr{H}_n(x)$ is replaced by the first term of its approximation with the cosine term omitted. The analysis applies equally well to the Laguerre case.

LEMMA 8. *Let p be a fixed number satisfying $1\leqq p\leqq 4/3$ or $4\leqq p\leqq\infty$, let $1/p+1/q=1$, let $0\leqq r<1$, let $v(x)$ be a measurable function and define*

$$g(y) = y^{-1/2}\left(\int_{ry}^{y-1}\left|\frac{v(x)}{(y-x)^{1/4}}\right|^p dx\right)^{1/p}\left(\int_{ry}^{y-1}\frac{dx}{|v(x)(y-x)^{1/4}|^q}\right)^{1/q}$$

with the usual interpretation if p or q is ∞. Then $\limsup_{y\to\infty} g(y)=\infty$.

By symmetry this need only be proved for $4\leqq p\leqq\infty$, and it can be assumed that $v(x)\geqq 0$. Let $s=r^{1/3}$. The proof will be done in two cases as indicated.

Case 1. For every number $z\geqq 1/(1-s)$ there exists $y\geqq z$ such that

$$\int_{s^2y}^{sy}[v(x)]^{-q}\,dx > z\int_{s^3y}^{s^2y}[v(x)]^{-q}\,dx. \tag{5.1}$$

Fix such a y and z. Since $sy=y-(1-s)y\leqq y-(1-s)z\leqq y-1$,

$$g(y) \geqq y^{-1/2}\left(\int_{s^3y}^{s^2y}\left[\frac{v(x)}{(y-x)^{1/4}}\right]^p dx\right)^{1/p}\left(\int_{s^2y}^{sy}\frac{dx}{[v(x)(y-x)^{1/4}]^q}\right)^{1/q}. \tag{5.2}$$

Now in (5.2) replace $y-x$ by y and use (5.1) on the second integral. This produces

$$g(y) \geqq \frac{z^{1/q}}{y}\left(\int_{s^3y}^{s^2y}[v(x)]^p\,dx\right)^{1/p}\left(\int_{s^3y}^{s^2y}[v(x)]^{-q}\,dx\right)^{1/q}. \tag{5.3}$$

Using Hölder's inequality on the product of the integrals in (5.3) then shows that $g(y)\geqq(s^2-s^3)z^{1/q}$. Since for an arbitrary z this inequality can be obtained for some

$y \geqq z$, $\limsup_{y\to\infty} g(y)=\infty$ in this case. Note that this reasoning works equally well for $p=\infty$.

Case 2. There exists a constant, C_1, such that if $y \geqq C_1$, then

$$\int_{s^2y}^{sy} [v(x)]^{-q}\,dx \leqq C_1 \int_{s^3y}^{s^2y} [v(x)]^{-q}\,dx. \tag{5.4}$$

Repeated use of (5.4) shows that there is a positive constant, C, such that if $y \geqq C_2=\max(C_1, 1/(1-s))$, then

$$\int_{ry}^{y-1} [v(x)]^{-q}\,dx \geqq C \int_{ry}^{y/s} [v(x)]^{-q}\,dx. \tag{5.5}$$

Now assume that $p<\infty$. Using (5.5) and the fact that $y-x<y$ for $ry \leqq x \leqq y-1$ then shows that

$$[g(y)]^p \geqq Cy^{1-p}\left(\int_{ry}^{y-1} \frac{[v(x)]^p}{y-x}\,dx\right)\left(\int_{ry}^{y/s} [v(x)]^{-q}\,dx\right)^{p/q}. \tag{5.6}$$

If $z \geqq C_2$ and $z \leqq y \leqq z/r$, (5.6) shows that $[g(y)]^p$ is bounded below by the product of

$$Cz^{1-p}\left(\int_{z}^{z/s} [v(x)]^{-q}\,dx\right)^{p/q} \tag{5.7}$$

and

$$\int_{ry}^{y-1} \frac{[v(x)]^p}{y-x}\,dx. \tag{5.8}$$

Using this fact and Fubini's theorem and then reducing the intervals of integration, shows that $\int_z^{z/r} [g(y)]^p\,dy$ is bounded below by the product of (5.7) and

$$\int_{z}^{z/s} [v(x)]^p\left(\int_{x+1}^{z/r} \frac{dy}{y-x}\right)dx. \tag{5.9}$$

Performing the inner integration then shows that (5.9) is bounded below by

$$C\log z(1-s)\int_{z}^{z/s} [v(x)]^p\,dx. \tag{5.10}$$

Therefore,

$$\int_{z}^{z/r} [g(y)]^p\,dy \geqq \frac{C\log Cz}{z^{p-1}}\left(\int_{z}^{z/s} [v(x)]^p\,dx\right)\left(\int_{z}^{z/s} [v(x)]^{-q}\,dx\right)^{p/q}. \tag{5.11}$$

By Hölder's inequality the product of the integrals in (5.11) is bounded below by $[z(1-s)/s]^p$ so that finally

$$\int_{z}^{z/r} [g(y)]^p\,dy \geqq Cz\log Cz, \tag{5.12}$$

where C depends only on r and p. Therefore, if $z \geqq C_2$, there is by (5.12) a value of y between z and z/r where $g(y) \geqq C(\log Cy)^{1/p}$. This completes Case 2 for $p<\infty$.

If $p=\infty$, then

$$g(y) = y^{-1/2}\left(\operatorname*{ess\,sup}_{[ry,y-1]} \frac{v(x)}{(y-x)^{1/4}}\right) \int_{ry}^{y-1} \frac{dx}{v(x)(y-x)^{1/4}}. \tag{5.13}$$

Now if $y \geqq C_3 = \max(C_1, 2/(1-s))$, then $ry \leqq y-2$ and using the fact that $y-x<y$ and (5.5) gives

$$g(y) \geqq Cy^{-3/4}\left(\operatorname*{ess\,sup}_{[y-2,y-1]} \frac{v(x)}{(y-x)^{1/4}}\right) \int_{ry}^{y/s} \frac{dx}{v(x)}. \tag{5.14}$$

If $z \geqq C_3$ and $z \leqq y \leqq z/r$, (5.14) shows that $g(y)$ is bounded below by the product of

$$Cz^{-3/4} \int_z^{z/s} \frac{dx}{v(x)} \tag{5.15}$$

and

$$\int_{y-2}^{y-1} v(x)\,dx. \tag{5.16}$$

Using this and Fubini's theorem and then reducing the intervals of integration, shows that $\int_z^{z/r} g(y)\,dy$ is bounded below by the product of (5.15) and

$$\int_{z-1}^{(z/r)-2} \left(\int_{x+1}^{x+2} dy\right) v(x)\,dx. \tag{5.17}$$

The condition $z \geqq 2/(1-s)$ implies that $(z/r)-2 \geqq z/s$. Therefore, (5.17) is bounded below by $\int_z^{z/s} v(x)\,dx$. Combining this with (5.15) then shows that

$$\int_z^{z/r} g(y)\,dy \geqq Cz^{-3/4}\left(\int_z^{z/s} \frac{dx}{v(x)}\right)\left(\int_z^{z/s} v(x)\,dx\right). \tag{5.18}$$

The product of the integrals in (5.18) is bounded below by

$$\left(\int_z^{z/s} [v(x)]^{-1/2}[v(x)]^{1/2}\,dx\right)^2 = z^2\left(\frac{1-s}{s}\right)^2$$

by Schwarz' inequality. This finally shows that for $z \geqq C_3$

$$\int_z^{z/r} g(y)\,dy \geqq Cz^{5/4}, \tag{5.19}$$

so that for a value of y between z and z/r, $g(y) \geqq Cy^{1/4}$. This completes Case 2 for $p=\infty$.

6. **Proof of Theorems 1 and 3.** It is now easy to combine the previous results to obtain these theorems. If there were a $w(x)$ of the type described in Theorem 1 that was not 0 almost everywhere, then Lemma 1 could be applied to obtain the

inequality (3.1) for some function $v(x)$. By Lemma 6 there exists y_0 such that if $y \geqq y_0$, $E_y = [\frac{3}{4}y^{1/2}, y^{1/2}-1]$ and I_y is the set of integers, n, such that $y \leqq 2n+1 \leqq y+y^{1/3}$, then for at least 3/5 of the n's in I_y

$$(6.1) \qquad \left(\int_{E_y} |\mathscr{H}_n(x)v(x)|^p \, dx\right)^{1/p} \geqq C\left(\int_{E_y} \left|\frac{v(x)}{(y-x^2)^{1/4}}\right|^p dx\right)^{1/p}$$

and for at least 3/5 of the n's in I_y

$$(6.2) \qquad \left(\int_{E_y} \left|\frac{\mathscr{H}_n(x)}{v(x)}\right|^q dx\right)^{1/q} \geqq C\left(\int_{E_y} \frac{dx}{|v(x)(y-x^2)^{1/4}|^q}\right)^{1/q}.$$

Consequently, there must be at least one n in I_y for which both (6.1) and (6.2) are true. Since for this n the product of the left sides of (6.1) and (6.2) is bounded by the right side of (3.1), the product of the right sides of (6.1) and (6.2) must be bounded by the constant C of Lemma 3. Since the constants are all independent of y, this shows that the product of the right sides of (6.1) and (6.2) is a bounded function of y for $y \geqq y_0$. Using the fact that $y^{1/2}+x \leqq 2y^{1/2}$ for $x \in E_y$ in these integrals then shows that there exists C such that for $y \geqq y_0$

$$(6.3) \qquad y^{-1/4}\left(\int_{E_y} \left|\frac{v(x)}{(y^{1/2}-x)^{1/4}}\right|^p dx\right)^{1/p}\left(\int_{E_y} \frac{dx}{|v(x)(y^{1/2}-x)^{1/4}|^q}\right)^{1/q} \leqq C.$$

This, however, is impossible by Lemma 8 for $1 \leqq p \leqq 4/3$ or $4 \leqq p \leqq \infty$.

For the proof of Theorem 3 the same reasoning using Lemma 7 shows that for $y \geqq y_0$

$$(6.4) \qquad \left(\int_{E_y} \left|\frac{v(x)}{x^{1/4}(y-x)^{1/4}}\right|^p dx\right)^{1/p}\left(\int_{E_y} \frac{dx}{|x^{1/4}v(x)(y-x)^{1/4}|^q}\right)^{1/q} \leqq C$$

where $E_y = [5y/6, y-y^{1/2}]$. Changing the variable of integration to $u = x^{1/2}$ changes the interval of integration in (6.4) to an interval containing $[19y^{1/2}/20, y^{1/2}-1]$. Using the fact that u and $y^{1/2}+u$ are both comparable to $y^{1/2}$ will then produce the inequality $g(y^{1/2}) \leqq C$ for $y \geqq y_0$ where g is the function in Lemma 8. As before, this is impossible for $1 \leqq p \leqq 4/3$ and $4 \leqq p \leqq \infty$.

7. **Proof of Theorems 2 and 4.** The principal difficulty in proving Theorem 2 is, as mentioned in §1, that it was not assumed that $w(x)f(x) \in L^p$ implies that $f(x)$ has a Hermite series. In the proof of Theorem 1 it was easy to obtain this fact from the hypotheses; this was done in the proof of Lemma 3. Here this seems difficult, and it is proved instead that $w(x)$ can be replaced by a larger and pleasanter function. The same will be done to prove Theorem 4.

First, several lemmas are needed.

LEMMA 9. *Given a fixed p, $1 \leqq p \leqq \infty$, there exists a positive function, $u(x)$, and a constant, C_1, such that $f(x)u(x) \in L^p$ on $(-\infty, \infty)$ implies that $f(x)$ has a Hermite series and $\|u(x)H_n(x)\|_p \leqq C_1 n^{-1/4}(2^n n!)^{1/2}$.*

LEMMA 10. *Given a fixed p and α, $1 \leqq p \leqq \infty$ and $\alpha > -1$, there exists a positive function $u(x)$ and a constant, C_1, such that $f(x)u(x) \in L^p$ on $(0, \infty)$ implies that $f(x)$ has a Laguerre series for this α and $\|u(x)L_n^\alpha(x)\|_p \leqq C_1 n^{\alpha/2-1/4}$.*

To prove Lemma 9 let $u(x) = \exp(-\frac{3}{4}x^2)$. If $\|f(x)u(x)\|_p < \infty$, it is easy to see that $\|f(x)H_n(x)e^{-x^2}\|_1$ is finite for every n by Hölder's inequality. The second part follows by using (2.1) and (2.6).

To prove Lemma 10 let $b = \frac{1}{2}(\alpha+1) - 1/p$ and define $u(x)$ to be x^b for $0 < x < 1$ and to be $\exp(-\frac{3}{4}x)$ for $x \geqq 1$. The first part then follows from Hölder's inequality and the fact that $(\alpha - b)p/(p-1) > -1$. The second part follows by use of (2.2), the fact obtained from Stirling's formula that $\Gamma(n+\alpha+1)/n! = n^\alpha[1+O(1/n)]$, and (2.13).

LEMMA 11. *If $1 \leqq p \leqq \infty$, I is a finite interval of length L and $w(x)$ is nonzero on a subset of I with positive measure, then there is a positive constant, C, such that for $a \geqq 1$ and all b $(\int_I |w(x) \cos(ax+b)|^p\, dx)^{1/p} \geqq C$.*

This follows immediately from the facts that $|w(x)|$ is bounded away from 0 on some subset of I with positive measure and that for any positive d, $|\cos(ax+b)|$ has a lower bound depending only on d and L on half of any subset of I with measure d.

LEMMA 12. *If $1 \leqq p \leqq \infty$ and $w(x)$ is nonzero on a set of positive measure, then there is a positive constant, C_2, such that $\|H_n(x)w(x)\|_p \geqq C_2 n^{-1/4}(2^n n!)^{1/2}$ for all $n \geqq 1$.*

LEMMA 13. *If $1 \leqq p \leqq \infty$, $\alpha > -1$ and $w(x)$ is nonzero on a subset of $(0, \infty)$ with positive measure, then there is a positive constant, C_2, such that $\|w(x)L_n^\alpha(x)\|_p \geqq C_2 n^{\alpha/2-1/4}$ for all $n \geqq 1$.*

To prove Lemma 12 let I be a finite interval such that $w(x)$ is nonzero on a subset of I with positive measure. Then using (2.5) and (2.1) and assuming that $w(x)$ is bounded in I shows that for some constant, C, $\|H_n(x)w(x)\|_p$ is bounded below by $(2/\pi)^{1/2}(\sqrt{\pi}2^n n!)^{1/2}N^{-1/4}$ times

$$\left(\int_I |\exp(x^2/2)w(x)\cos(N^{1/2}x - \tfrac{1}{2}n\pi)|^p\, dx\right)^{1/p} - CN^{-1/2}\left(\int_I |\exp(x^2/2)w(x)|^p\, dx\right)^{1/p} \tag{7.1}$$

where $N = 2n+1$. Using Lemma 11, it is immediate that (7.1) is bounded below by a constant if n is greater than some n_0 so that Lemma 12 is proved if $w(x)$ is bounded and $n \geqq n_0$. The statement of Lemma 12 shows that the restriction that $w(x)$ is bounded can be dropped, and since $\|H_n(x)w(x)\|_p$ is positive for all n, a C_2 can be chosen that works for all $n \geqq 1$.

The proof of Lemma 13 uses (2.12) and (2.2) and is the same except that to use Lemma 11 on the analogue of (7.1) a change of variable must be made.

It is now easy to complete the proof of Theorem 2. Suppose that there were a $w(x)$ that satisfied the hypotheses of Theorem 2 and was nonzero on a set of positive measure. Then let $w^*(x) = |w(x)| + C_2 u(x)/C_1$ where C_1 and $u(x)$ are as in Lemma 9 and C_2 is as in Lemma 12 with this $w(x)$. Then using Lemmas 9 and 12 and Minkowski's inequality, it is clear that

$$\|w^*(x)H_n(x)\|_p \leqq 2\|w(x)H_n(x)\|_p. \tag{7.2}$$

Now if $\|w^*(x)f(x)\|_p < \infty$, then $\|w(x)f(x)\|_p < \infty$ and by (7.2) and the hypothesis of Theorem 2 it follows that $\lim_{n\to\infty} \|a_n H_n(x) w^*(x)\|_p = 0$. Therefore, $w^*(x)$ satisfies the hypotheses of Theorem 2. Since $w^*(x)$ is bounded below by a positive multiple of $u(x)$, Lemma 9 shows that $\|w^*(x)f(x)\|_p < \infty$ implies that $f(x)$ has a Hermite series. Then the Banach-Steinhaus theorem and Theorem 1 show that $w^*(x) = 0$ almost everywhere. Since $|w(x)| \leqq w^*(x)$, $w(x) = 0$ almost everywhere. This contradiction completes the proof of Theorem 2.

The proof of Theorem 4 is identical using Lemmas 10 and 13.

References

1. R. Askey and S. Wainger, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. **87** (1965), 695–708. MR **32** #316.

2. A. Erdélyi, *Asymptotic solutions of differential equations with transition points or singularities*, J. Mathematical Phys. **1** (1960), 16–26. MR **22** #2773.

3. ———, *Asymptotic forms for Laguerre polynomials*, J. Indian Math. Soc. **24** (1960), 235–250. MR **23** #A1073.

4. H. Pollard, *The mean convergence of orthogonal series*. II, Trans. Amer. Math. Soc. **63** (1948), 355–367. MR **9**, 426.

5. G. Szegö, *Orthogonal polynomials*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R. I., 1959. MR **21** #5029.

6. A. Zygmund, *Trigonometric series*. Vols. I, II, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR **21** #6498.

Rutgers University,
New Brunswick, New Jersey
The Institute for Advanced Study,
Princeton, New Jersey